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INVARIANT IMBEDDING AND NEUTRON TRANSPORT THEORY-III.
NEUTRON-NEUTRON COLLISION PROCESSES

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SUMMARY

The effects on criticality of neutron-neutron collisions involving annihilation are investigated for one-dimensional, single and multi-group cases. The analytic treatment shows that regardless of the magnitude of the cross section for collision between moving neutrons, there is no critical length (mass).

The analogy between this situation and that in hydrodynamics, where the addition of an arbitrarily small viscosity term eliminates the discontinuous shock phenomenon, is indicated.

As in earlier papers, the underlying equations are derived using the principle of invariant imbedding.

INVARIANT IMBEDDING AND NEUTRON TRANSPORT THEORY-III.

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by

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1. Introduction

In previous papers in this series, [2,3,4], we have considered neutron transport models in which various types of collisions (scattering, absorption, fission) were allowed to occur between the moving particles and fixed nuclei. The effects of collisions between neutrons were not considered. In the current investigation we include this process, as well as the others, assuming that the new type of collision results in the annihilation of the particles involved.

Considerations are again confined to a one-dimensional model, though particles with several possible energy states are included. The concept of invariant imbedding, [1], is used to derive the functional equations which constitute a mathematical description of the physical processes.

First, the internal flux equations are derived, assuming a very general and physically reasonable neutron-neutron collision law. A

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special case is then investigated in some detail. The equations so obtained are amenable to analysis, although not easily solved explicitly. It is worthwhile, then, to present the results of some numerical experimentation.

The analytic treatment shows, regardless of the magnitude of the cross section for collision between moving neutrons, there is no critical length (mass) for the rod. Arbitrarily large fluxes may be obtained, as might be imagined, if the source is strong enough.

The analogy between this situation and that in hydrodynamics, where the addition of an arbitrarily small viscosity term eliminates the discontinuous shock phenomenon, is then pointed out.

Various interconnections among internal fluxes and transmitted and reflected fluxes are exhibited for the models considered. They are, of course, more complicated than those described in [4], where the effects of collisions between moving particles were neglected.

2. A Collision Model

Consider a rod of length x containing nuclei which are fixed in position. A neutron moving in the rod may collide with a nucleus, which results in one of several possible events:

a. the neutron may be absorbed without creation of more particles;

b. it may be scattered in the forward or backward direction;

c. it may disappear itself but give rise to new neutrons through the process of fission.

For convenience we assume, initially, that all neutrons are at the same energy level. Thus the events described may be aggregated by

assuming that a neutron making a collision with a nucleus gives rise, on the average, to F neutrons moving in the original direction of travel and B neutrons moving in the opposite direction. Steady state conditions are assumed in deriving the basic equations.

In addition, we shall suppose that neutrons moving in opposite directions may collide with one another, resulting in their annihilation. To reduce these ideas to mathematical form let us introduce some convenient notation.

Assume that y neutrons per unit time are introduced into the system at x, as is shown in the figure below.

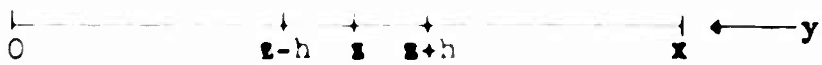

Figure 1. The Physical Situation

Let

(1) $\sigma h + o(h)$ = probability that a neutron will collide with a nucleus in a segment of length h. (Here σ is the cross section and is the same as λ^{-1} used in [4].)

(2) $u(z;x,y)$ = the expected number of neutrons per unit time passing an interior point z in the direction opposed to that of the incident neutrons.

(3) $v(z;x,y)$ = the expected number of neutrons per unit time passing an interior point z in the same direction as the incident neutrons.

(4) $k(u,v)h + o(h)$ = the expected number of neutrons in a stream of strength u which are annihilated per unit time due to collisions with an opposing stream of strength v, in an interval of length h.

Though we shall not enter into a detailed discussion of the desirable properties of the collision function $k(u,v)$, clearly we should expect

(5) (a) $k(u,v) = k(v,u)$,

(b) $k(u,0) = 0$,

(c) $k(1,1) = b$,

where b is the effective cross section for collision between neutrons. The symmetric behavior in (5a) is convenient, but not necessary.

3. <u>The</u> <u>Internal</u> <u>Flux</u> <u>Equations</u>

By noting the expected flows past observers at z and $z-h$, after suppressing the dependence of the functions u and v on x and y, we see that

(1) $u(z) = u(z-h)(1-\sigma h) + u(z-h)\sigma hF + v(z)\sigma hB - hk(u,v) + o(h)$

Similarly for the function $v(z)$ we find the relation

(2) $v(z) = v(z+h)(1-\sigma h) + v(z)\sigma hF + u(z)\sigma hB - hk(u,v) + o(h)$

Letting $h \to 0$, we are lead to the nonlinear system of differential equations for the internal fluxes

(3) $u'(z) = (F-1)\sigma u(z) + B\sigma v(z) - k(u(z),v(z))$,

$v'(z) = (1-F)\sigma v(z) - B\sigma u(z) + k(u(z),v(z))$.

The boundary conditions are

(4) $u(0) = c$, $v(x) = y$,

which follow from the formulation. Observe that these are two-point conditions, rather than initial values at 0 or at x.

In the following discussion, we shall assume that the collision function $k(u,v)$ is given by

$$(5) \qquad k(u,v) = buv, \quad b > 0,$$

and that as a result of a fissioning one neutron is forward-scattered and one is backward-scattered, so that

$$(6) \qquad F = B = 1.$$

Under these assumptions the equations in (3) reduce to

$$(7) \qquad \begin{aligned} u' &= \sigma v - buv, \\ v' &= -\sigma u + buv. \end{aligned}$$

Along with the boundary conditions of equation (4), the equations (7) represent a two-point nonlinear boundary value problem for the expected values of the internal fluxes.

4. Analysis and Computational Results

Though, in general, the discussion of a two point nonlinear boundary value problem presents formidable difficulties from both the analytical and computational viewpoints, in this case it is possible to proceed in rather straightforward fashion. We consider solution curves which pass through the point

$$(1) \qquad v(0) = d, \quad u(0) = 0,$$

for $z \geq 0$.

It is convenient to normalize the unit of distance in such a way that

(2) $\sigma = 1,$

and to consider three cases, depending on whether d, as defined in equation (1) is less than, equal to, or greater than b^{-1}.

We first consider that

(3) $0 < d < b^{-1}.$

From equation (3.7),

(4) $u' = v - buv = v(1-bu)$

$v' = -u + buv = u(bv-1),$

we see that initially u is increasing and v is decreasing. As z increases, v decreases and u increases, so that u' decreases and v' becomes more and more negative. Eventually v must become zero at, say, the value

(5) $z = z_1 = z_1(d),$

at which point we have

(6) $u'(z_1) = 0.$

Let us now show that

(7) $0 \le u(z) \le b^{-1}, \qquad 0 \le z \le z_1.$

Upon integrating the equation

$$(8)\qquad \frac{du}{dv} = \frac{v(1-bu)}{u(bv-1)},$$

which follows from equation (4), we find the relation

$$(9)\qquad \log \frac{(1-bu)(1-bv)}{1-bd} = b(d-v-u),$$

where

$$(10)\qquad 0 \leq v \leq d < b^{-1}.$$

Consequently in this case the curve $u = u(z)$ cannot cross the line $u = b^{-1}$. In addition from formula (9) we determine that

$$(11)\qquad v(0) = d = u(z_1) = u_{max},$$

provided $d < b^{-1}$.

We have now established that both $u(z)$ and $v(z)$ are monotone and uniformly bounded by b^{-1} on the interval $[0, z_1(d)]$, for $d < b^{-1}$.

Graphs for this case in which $b = .01$ and various values of d are assigned are shown in Figure 2. The curves were obtained by Dr. E. C. DeLand using an analogue computing machine.

The case in which $d = b^{-1}$ can be resolved explicitly. The result is

$$(12)\qquad u(z) = b^{-1}(1-e^{-z}),$$

$$v(z) = b^{-1}.$$

If $d > b^{-1}$, then initially u and v increase as z increases. The function $v'(z)$ also increases. On the other hand, since $u(z)$ cannot cross the line $u = b^{-1}$, and u is monotone increasing, u approaches a limit, which, according to equation (9) must be b^{-1}. Graphs for this case with $b = .01$ are shown in Figure 3.

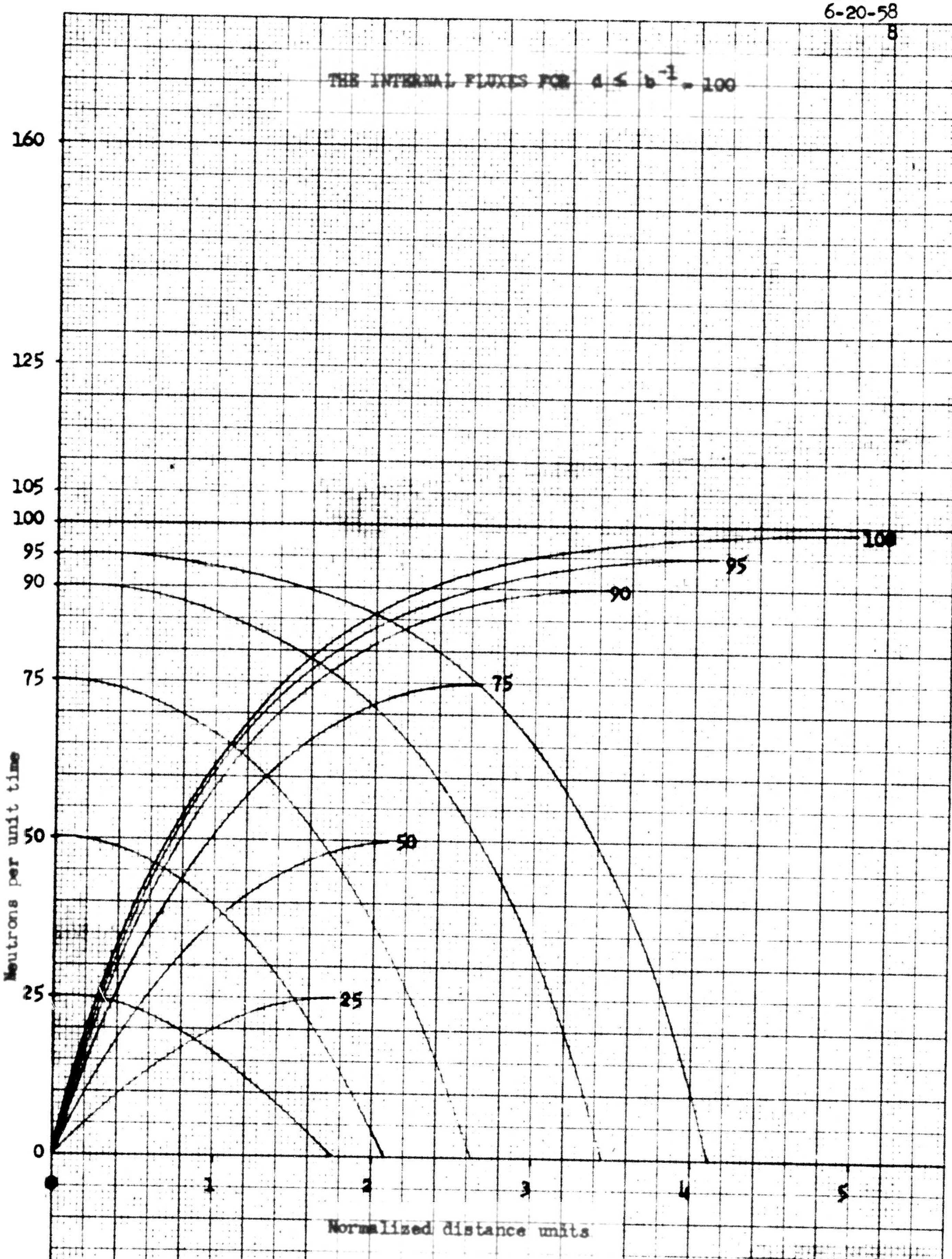
THE INTERNAL FLUXES FOR d ≤ b-1 = 100
160
125
105
100
95
90
75
50
25
0
100
95
90
75
50
25
1
2
3
4
5
Neutrons per unit time
Normalized distance units

FIG. 2

It is a straightforward matter to prove that for each x, $y>0$ there exists a v-curve passing through the point (x,y), as well as a corresponding u-curve passing through the point $(0,0)$. A proof by contradiction is readily constructed.

In summary we state that regardless of the value of $v(0) = d$,

$$0 \le u(z) \le b^{-1}, \quad 0 \le z \le z_1(d). \tag{13}$$

For the function $v(z)$ we have

$$0 \le v(z) \le \text{Max}\left[v(x), b^{-1}\right], \quad 0 \le z \le x.$$

Lastly, we note that if a source of neutrons is applied to a sufficiently long rod, then, under the assumptions we have made, both the number of neutrons reflected and the number of neutrons transmitted are approximately equal to b^{-1}. This is the asymptotic value as $x \to \infty$.

The physical meaning of this, in marked contrast to the case in which no annihilation of neutrons through neutron-neutron collision takes place, [4], is that there is no critical length (mass) of the rod. The internal flux which in direction is opposed to the incident flux at each interior point of the rod is bounded <u>a priori</u> in terms of the collision coefficient. The internal flux which in direction agrees with that of the incident flux is bounded by an expression depending on the coefficient of collision and the source strength, and may be made arbitrarily large by having a sufficiently strong source. These results must not be taken too seriously in any physical situation, since the assumption that the interaction between u and v has the form buv may very well break down as u and v become large.

THE INTERNAL FLUXES FOR $d \geq h^{-1} = 100$

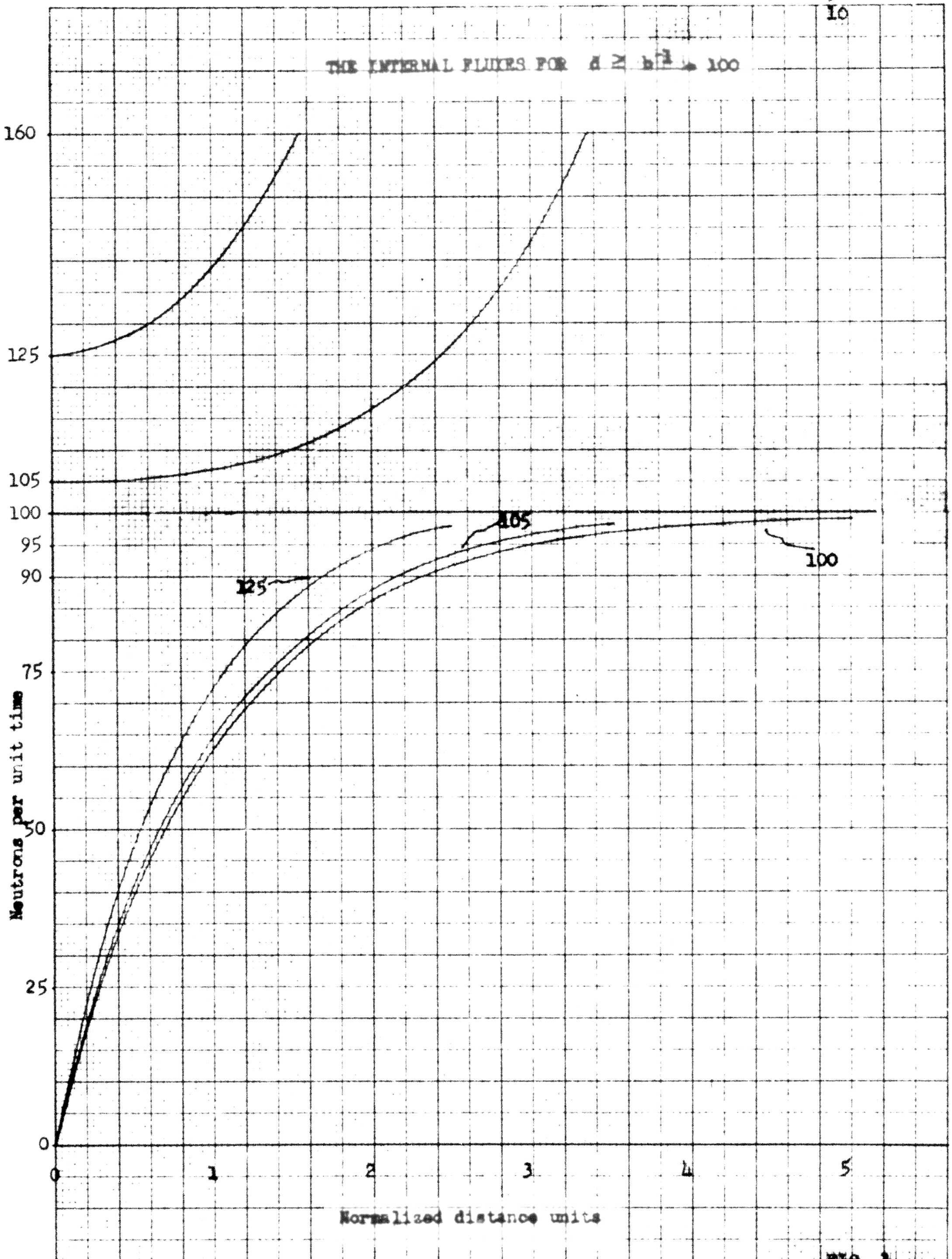

FIG. 3

5. Perturbation Considerations

An interesting example of the difficulties which can arise through formal use of perturbation procedures can now be given. Once again consider the system

$$(1) \qquad \begin{aligned} u' &= v(1-bu), \\ v' &= u(bv-1), \qquad 0 \le z \le x, \end{aligned}$$

along with the boundary conditions

$$(2) \qquad u(0) = 0, \quad v(x) = y,$$

where b is considered to be a small quantity. Let us put, following the usual procedure,

$$(3) \qquad \begin{aligned} u &= u_0 + bu_1 + b^2u_2 + \dots , \\ v &= v_0 + bv_1 + b^2v_2 + \dots , \end{aligned}$$

where

$$(4) \qquad \begin{aligned} &u_i(0) = 0, \quad i = 0, 1, 2, \dots , \\ &v_i(x) = \begin{cases} y, & i = 0, \\ 0, & i = 1, 2, 3, \dots . \end{cases} \end{aligned}$$

For u_o and v_o we obtain the equations

$$(5) \qquad \begin{aligned} &u_o' = v_o, \quad u_o(0) = 0, \\ &v_o' = -u_o, \quad v_o(x) = y, \end{aligned}$$

for which the solution is readily seen to be

$$u_o(z) = \frac{y}{\cos x} \sin z,$$

(6)

$$v_o(z) = \frac{y}{\cos x} \cos z, \qquad 0 \le z \le x < \frac{\pi}{2} .$$

A critical length exists and is $\frac{\pi}{2}$, since both u_o and v_o become infinite as x, the length of the rod, is increased to this length. This agrees with the results previously obtained in [2] for the linear case.

The equations for the function $u_1(z)$ and $v_1(z)$ are

$$u_1' = v_1 - u_o v_o, \qquad u_1(0) = 0,$$

(7)

$$v_1' = -u_1 + u_o v_o, \qquad v_1(x) = 0.$$

The solution is of the form

$$u_1 = \frac{y^2}{\cos^2 x} \quad f_1(z,x),$$

(8)

$$v_1 = \frac{y^2}{\cos^2 x} \quad f_2(z,x), \qquad 0 \le z \le x < \frac{\pi}{2} \quad ,$$

where the functions f_1 and f_2 are bounded away from zero as x tends to $\frac{\pi}{2}$ for almost all z.

This would tend to strengthen the belief that $\frac{\pi}{2}$ is the critical length. In reality, as we know from Section 4, there is no critical length. The transition from the case $b = 0$ to the case $b > 0$ corresponds to a drastic change in the nature of the solutions with regard to the existence of criticality. We would, of course, be warned of this by the fact that the supposed perturbation terms, u_1 and v_1, are actually of larger magnitude than u_o and v_o as $x \longrightarrow \frac{\pi}{2}$.

7. The Reflected and Transmitted Fluxes

Let us now consider the problem of determining directly the reflected and transmitted fluxes under the same assumptions of Sections 3 and 4. In particular we seek the reflected and transmitted fluxes from a homogeneous bar of length x with an incident flux y. In the spirit of the principle of invariant imbedding, [1], we imbed this problem within the class of problems of determining these fluxes for bars of all lengths $x \geq 0$. The problem is trivial for $x = 0$, and knowledge of the solution for a bar of length x enables us to determine the solution for a bar of length $x + h$.

Mathematically, we are led to the reflection and transmission functions as solutions of initial value problems. Knowledge of these functions then enables us to determine the internal fluxes, $u(z)$ and $v(z)$, as solutions of initial value problems, which is of great importance from the computational viewpoint.

We introduce the function

(1) $r(y;x)$ = the expected number of neutrons reflected per unit of time from a homogeneous bar of length x as a result of having y incident neutrons per unit of time.

Figure 4. The Reflected Flux.

The expected number of neutrons reflected from a bar of length x+h is the sum of three terms, to within terms with probabilities of orders zero and one in h. A neutron upon passing through the segment $[x+h, x]$ may undergo fission in that segment. If it does not, it will give rise to some neutrons which will be reflected from the rod of length x. In turn some of these will enter the flux reflected from the bar of length x+h and some will undergo fission in the segment $[x, x+h]$ thus giving rise to neutrons which re-enter the bar of length x, ultimately to contribute to the total flux reflected from the bar of length x+h. All other processes which give rise to the reflected flux have probabilities that are of order greater than the first and so may be neglected, as will be seen.

These considerations lead to the equation

$$(2)\quad r(y;x+h) = y\sigma h + r(y-bhr(y;x);x)\left[1-byh\right] + r(\sigma hr(y;x);x) + o(h)\ .$$

By letting h tend to zero and assuming $r(0;x)$ vanishes we find that $r(y;x)$ satisfies the quasilinear first order partial differential equation

$$(3)\quad r_x(y;x) = \sigma y - br(y;x)r_y(y;x) - byr(y;x) + \sigma r(y;x)r_y(x,0),\quad 0\le y,\ x,$$

where, as usual, the subscripts indicate partial differentiation. The reflection function $r(y;x)$ also satisfies the initial condition

$$(4)\qquad r(y;0) = 0.$$

The equation (3) specializes, for b = 0, to the Riccati equation derived in our earlier papers for the reflection coefficient. It may be resolved via characteristic theory, [6], or by direct numerical

integration, returning essentially to (2). The equations for the characteristics are

$$\frac{dx}{ds} = 1$$

(5) $$\frac{dy}{ds} = br$$

$$\frac{dr}{ds} = \sigma y - byr + \sigma r\, r_y(x;0).$$

Since $x=s$, $y=0$, $r=0$ is a solution of the system (5) passing through the point $x=y=r=0$, we find that

(6) $$r(0;x) = 0,$$

as was assumed above on physical grounds.

Once the function $r(y;x)$ has been determined for suitable ranges of y and x, one may reduce the determination of $u(z)$ and $v(z)$, the internal fluxes, to the solution of initial value problems, as was mentioned earlier. If the incident flux $v(x) = y$ is specified, then the reflected flux is $r(y;x) = u(x)$, so that now both $u(z)$ and $v(z)$ are specified at $z = x$. Through use of equations (3.7) the functions $u(z)$ and $v(z)$ may now be determined on the entire interval $[0,x]$.

The equations satisfied by the transmitted flux $t(y;x)$, where

(7) $t(y;x)$ = the expected number of neutrons emergent from the end $z = 0$ of a homogeneous bar of length x as a result of having y neutrons per unit time incident on the end $z = x$,

are similarly derived.

We have

(8) $t_x(y;x) = -br(y;x)\ t_y(y;x) + \sigma r(y;x)\ t_y(0;x), \qquad 0 \le y,x,$

along with the boundary conditions

(9) $t(0;x) = 0, \quad t(y;0) = y.$

7. <u>Internal</u> <u>Sources</u>

Investigations paralleling those in [4] can be carried through for the determination of emergent fluxes due to internal sources. Let, for example,

(1) $w(y;z;x)$ = the expected number of neutrons emerging per unit time from the x-end of a bar of length x as a result of a source of strength y neutrons per unit time, moving toward x, at the point z.

We find that

(2) $$w(y;z;x+h) = w(y;z;x) + r(wah;x) + o(h)\ .$$

This leads to the system

(3) $$w_x = ar_y(0;x)w, \qquad x \ge z,$$

$$w(y;z;z) = y,$$

for which the solution is

(4) $$w = y \exp\left\{a \int_z^x r_y(0;s)ds\right\}\ .$$

8. Two-Group Theory

Following the usual approximation to the physical situation where a neutron possesses a direction, and an energy which varies continuously between certain limits, let us assume that there are just two types of neutrons, 'fast' and 'slow,' and that in the fission process either can give rise to the other. In addition, either type can annihilate the other in a neutron-neutron collision.

To simplify the equations let us assume that fast neutrons have a probability $\sigma_F h + o(h)$ of splitting in an interval of length h and that when they do split there is probability one-half of a fast neutron produced going in one direction and a slow neutron going in the other, and probability one-half of a slow neutron going in the former direction and a fast neutron going in the latter direction. The same situation is to prevail for slow neutrons with σ_F replaced by σ_S. To account for the annihilation of neutrons through collision let us assume that the expected number of fast neutrons annihilated per unit time in an interval of length h as a result of collisions with an opposing stream of fast neutrons is $b_{FF} u_F(z) v_F(z) h + o(h)$, where $u_F(z)$ and $v_F(z)$ are the respective stream strengths. The other collision coefficients b_{FS} and b_{SS} are defined similarly, as are the functions $u_S(z)$ and $v_S(z)$. Collisions between slow neutrons and overtaking fast neutrons are neglected. We specify that y_F fast neutrons per unit time are incident on the bar at $z = x$, as are y_S slow neutrons per unit time.

Taking into account the various fission and collision processes which can occur in an interval of length h we find for the function $u_F(z)$

$$u_F(z+h) = (1-\sigma_F h)\, u_F(z) + \frac{1}{2}\sigma_F h\left[u_F(z) + v_F(z)\right] +$$

(1) $$\frac{1}{2}\sigma_S h\left[u_S(z) + v_S(z)\right] - b_{FF} h u_F(z)\, v_F(z)$$

$$- b_{FS}\, h u_F(z)\, v_S(z) + o(h).$$

Passing to the limit by letting h tend to zero we find

(2) $$u_F' = \frac{1}{2}\sigma_F(v_F - u_F) + \frac{1}{2}\sigma_S(u_S + v_S) - b_{FF}u_F v_F - b_{FS}u_F v_S.$$

As a boundary condition for the function $u_F(z)$ we have

(3) $$u_F(0) = 0.$$

Similar considerations then yield the following nonlinear system and boundary conditions for the functions $u_S(z)$, $v_F(z)$, $v_S(z)$:

$$u_S' = \frac{1}{2}\sigma_S(v_S - u_S) + \frac{1}{2}\sigma_F(u_F + v_F) - b_{FS}u_S v_F - b_{SS}u_S v_S,$$

(4) $$v_F' = \frac{1}{2}\sigma_F(u_F - v_F) + \frac{1}{2}\sigma_S(u_S + v_S) - b_{FS}v_F u_S - b_{FF}v_F u_F,$$

$$v_S' = \frac{1}{2}\sigma_S(u_S - v_S) + \frac{1}{2}\sigma_F(u_F + v_F) - b_{FS}v_S u_F - b_{SS}v_S u_S,$$

(5) $u_S(0) = 0, \quad v_F(x) = y_F, \quad v_S(x) = y_S.$

The resolution of the nonlinear two-point boundary value problem of equations (2), (3), (4) and (5) is troublesome, even when attempted numerically on a high speed computing machine. For situations involving more than two velocity groups this problem becomes even more serious.

It is advantageous, therefore, to undertake the analysis from a different viewpoint, viz., the determination of reflected and transmitted fluxes for rods of length $x \geq 0$, which leads to initial value problems.

Let us introduce the functions $r_F(y_F, y_S; x)$ and $r_S(y_F, y_S; x)$ defined to be

(6) $r_F(y_F, y_S; x)$ = the expected number of fast neutrons reflected per unit time from a homogeneous rod of length x as a result of y_F fast neutrons and y_S slow neutrons incident per unit time on the end $z = x$.

(7) $r_S(y_F, y_S; x)$ = the corresponding quantity for the slow neutrons reflected.

Through reasoning paralleling that of Section 6 we derive the following equation for the function r_F:

$$r_F(y_F, y_S; x+h) = \frac{1}{2}\sigma_F h y_F + \frac{1}{2}\sigma_S h y_S + \tag{8}$$

$$\left[1 - \frac{1}{2}\sigma_F h - b_{FS} y_S h - b_{FF} y_F h\right] r_F(y_F - b_{FF} r_F y_F h - b_{FS} r_S y_F - \frac{1}{2}\sigma_F h y_F + \frac{1}{2}\sigma_S h y_S, y_S - \cdots; x)$$

$$+ \frac{1}{2}\sigma_S h r_S(y_F, y_S; x)$$

$$+ r_F\left(\frac{1}{2}\sigma_F h r_F + \frac{1}{2}\sigma_S h r_S, \frac{1}{2}\sigma_F h r_F + \frac{1}{2}\sigma_S h r_S; x\right) + o(h).$$

The corresponding equation for the function r_S is

$$(9) \quad r_S(y_F, y_S;\ x+h) = \frac{1}{2}\ \sigma_F h y_F + \frac{1}{2}\ \sigma_S h y_S +$$

$$\left[1 - \frac{1}{2}\ \sigma_S h - b_{FS} y_F h - b_{SS} y_S h\right]\ r_S(y_F - b_{FF} r_F y_F h - b_{FS} r_S y_F h$$

$$- \frac{1}{2}\ \sigma_F h y_F + \frac{1}{2}\ \sigma_S h y_S,\ y_S - \ldots;\ x)$$

$$+ \frac{1}{2}\ \sigma_F h r_F(y_F, y_S; x) +$$

$$r_S(\frac{1}{2}\ \sigma_F h r_F + \frac{1}{2}\ \sigma_S h r_S,\ \frac{1}{2}\ \sigma_F h r_F + \frac{1}{2}\ \sigma_S h r_S; x) + o(h).$$

By passage to the limit in equations (8) and (9) we obtain a nonlinear system of first-order partial differential equations for the reflection functions r_F and r_S. The initial conditions are

$$(10) \quad r_F(y_F, y_S; 0) = r_S(y_F, y_S; 0) = 0.$$

Correspondingly, equations for the transmitted fluxes can be derived.

As in the one-velocity case, once r_F and r_S have been determined for suitable ranges of the independent variables, the determination of the internal fluxes is reduced to an initial value problem.

9. Analogy between Shock Waves and Critical Mass

In our paper, [3], we derive the quasilinear partial differential equation

$$(1) \qquad u_x = fu_t - h(u + tu_t) + atu(1 + u_t) + cuu_t + et + f$$

for the generating function associated with the number of neutrons

reflected from a one-dimensional case.

This equation is a generalized version of the equation

$$(2) \qquad u_t + uu_x = 1$$

(Courant-Hilbert, V. II, p. 55), which is used to illustrate the nature and origin of one-dimensional shock waves.

Comparing the two equations, we observe an interesting analogy between the time at which a shock first occurs and the length at which criticality occurs.

In the present paper, this analogy is pushed further. The equation of Burgers, [5,7]

$$(3) \qquad u_t + uu_x = bu_{xx}, \qquad b > 0,$$

where the term bu_{xx} corresponds to a physical viscosity, does not exhibit a shock. Similarly, the introduction of a collision phenomenon eliminates criticality.

The interesting thing to do is to examine the corresponding situation for multi-dimensional shocks and multi-dimensional fission processes, and this we shall do in a subsequent publication.

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